0913 Lecture 03

Tuesday, September 06, 2016 1:21 AM

[FIRST DISCUSS FUNCTOR OF POINTS FORMALISM]

Group actions on schemes:

Def: $G \times X \xrightarrow{\sigma} X$ satisfying axioms $\begin{cases} 1) \text{ identity} \\ 2) \text{ associativity} \end{cases}$

X a scheme

ex1: $X \to G \times X \to X$

Important map $G \times X \longrightarrow X \times X$
$(g, x) \longmapsto (gx, x)$

(or more generally $\psi_f : G \times T \to X \times T$ for any T-point $f: T \to X$)

e.g. for a k-point, get $\psi_x : G \to X \rightsquigarrow$ its image is the orbit, fiber over x is $\mathrm{Stab}_x \subset G$, a sub gp

↳ more generally

$$\begin{tikzcd} \mathrm{Stab}_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G \times T \arrow[d, "\psi_f"] \\ T \arrow[r, "{(f,1)}", "\mathrm{diag}"'] & X \times T \end{tikzcd}$$

get inertia group, $I_X \rightsquigarrow$ a group scheme over X giving stabilizer of each point

First observations:

- Fiber dimension of $I_X \longrightarrow X$ is upper semi-continuous because it is a locally finite type group scheme
- Fiber at a point is "stabilizer" image is

- Fiber at a point is "stabilizer" $=G_x$, image is "orbit"
- If X Noetherian, $\dim(G) = \dim(G_x) + \dim(G\cdot x)$ (fin. Krull dim)
- Closure $\overline{G\cdot x}$ is a union of lower dim. orbits, contains a closed orbit (equiv. to minimal)

Most concrete: Actions on affine scheme $X = \operatorname{Spec} A$

Equivalent to giving $k[G]$-comodule structure on A such that $\rho : A \to A \otimes k[G]$ is a map of algebras

Lemma: If A is finite type$/k$, there is a linear rep. V and an embedding $\operatorname{Spec} A \hookrightarrow V$ which is G-equivariant

Pf (idea): V is the dual of a finite dim'l invariant sub-rep of A which generates A as an algebra. $\square$

Remark: Matsushima's theorem: If G is a reductive algebraic group and $H \subset G$ an algebraic subgroup, then

H reductive $\Leftrightarrow$ G/H is affine

H reductive $\iff$ G/H is affine
↳ always q-proj.

(Alper has proves the same result for lin. reductive fppf group schemes over arbitrary base)

Then choose $G \hookrightarrow GL_n$, then GL_n acts transitively on $X = GL_n/G = \operatorname{Spec} A$
by above, have closed imm. $\operatorname{Spec}(A) \hookrightarrow V$ for some GL_n-rep V
$\implies$ $\forall$ reductive G, $\exists$ a rep of GL_n and a $v \in V$ w/ closed orbit such that $G = \operatorname{stab}(v)$

This statement (except for v having closed orbit) is true for any G.

Eg: an affine scheme w/ T action, where T is split torus, is equiv. to an M-graded algebra, where M is char lattice
↳ $\boxed{\text{useful fact:}}$ every invariant ideal is gen. by homogeneous elements.

Next level of complexity: V linear rep of G $\rightsquigarrow$ action on $\mathbb{P}(V)$

Consider $X \hookrightarrow \mathbb{P}(V)$ which is equivariant ($G \times X \longrightarrow \mathbb{P}(V)$ factors through X, in which case it does so uniquely, and induces an action on X)

(Rem: we will see later that this is true for any normal proj. variety)

Special results for T-varieties:

Thm (Sumihiro) Let $X \hookrightarrow \mathbb{P}(V)$ be a T-equivariant q-proj. subvariety, then for any point $x \in X$, $\exists$ a T-equivariant affine open containing x.

Proof: Case 1: T splits, and $X \hookrightarrow \mathbb{P}(V)$ closed: use standard open $\mathbb{P}(V)_f \cap X$ where $f \in V^*$ is eigenvect.

Case 2: T is split, and $X \hookrightarrow \mathbb{P}(V)$ loc. closed: consider closure, then have T-inv. affine variety $Y \supset X \cap Y \ni x$

Show that you can find an eigenvector $f \in \mathcal{O}(Y)$ which vanishes on $Y \setminus X \cap Y$ and does not vanish at x

General case: T splits after some finite Galois extension k'/k. Choose a $T_{k'}$-equivariant affine open $U \subset X_{k'}$ $\rightsquigarrow$ because X is separated,

$U = \bigcap_{\sigma \in Gal(k'/k)} \sigma(U)$ is a $T_{k'}$-equiv. aff. open which descends

$U = \bigcap_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U)$ is a $T_{k'}$ equiv. aff. open, which descends to a T-equiv. aff. open of X

$\blacksquare$

[INSERT DISCUSSION OF FIXED POINTS AND BIALYNICKI-BIRULA THEOREM]

[INSERT DISCUSSION OF QUOTIENTS]