

[FIRST DISCUSS FUNCTOR OF POINTS FORMALISM]

Group actions on schemes:

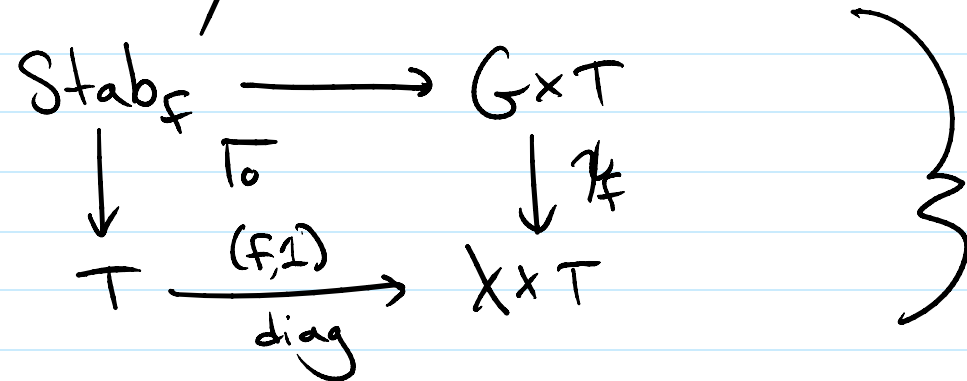
Def: $G \times X \xrightarrow{\sigma} X$ satisfying
 X a scheme

axioms $\left\{ \begin{array}{l} 1) \text{ identity} \\ 2) \text{ associativity} \end{array} \right.$ $X \xrightarrow{\text{ex1}} G \times X \rightarrow X$

Important map $G \times X \rightarrow X \times X$
 $(g, x) \mapsto (gx, x)$ (or more generally $\psi_f: G \times T \rightarrow X \times T$
 for any T -point $f: T \rightarrow X$)

e.g. for a k -point, get orbit, fiber over x is $\psi_x: G \rightarrow X \rightsquigarrow$ its image is the $\text{Stab}_x \subset G$, a sub gp

\hookrightarrow more generally



} get inertia group, $I_x \rightsquigarrow$ a group scheme over X giving stabilizer of each point

First observations:

- Fiber dimension of $I_x \rightarrow X$ is upper semi-continuous because it is a locally finite type group scheme
- Fiber at a point is "stabilizer" image is Stab_x

- Fiber at a point x is "orbit" $G \cdot x$ is "stabilizer" G_x image is $G \cdot x$
- If X Noetherian, $\dim(G) = \dim(G_x) + \dim(G \cdot x)$ (Fin. Krull dim)
- Closure $\overline{G \cdot x}$ is a union of lower dim. orbits, contains a closed orbit (equiv. to minimal)

Most concrete: Actions on affine scheme $X = \text{Spec } A$

Equivalent to giving $k[G]$ -comodule structure on A such that $\rho: A \rightarrow A \otimes k[G]$ is a map of algebras

Lemma: If A is finite type $/k$, there is a linear rep. V and an embedding $\text{Spec } A \hookrightarrow V$ which is G -equivariant

PF (idea) V is the dual of a finite dim k invariant sub-rep of A which generates A as an algebra

Remark: Matsushima's theorem: If G is a reductive algebraic group and $H \subset G$ an algebraic subgroup, then H reductive $\iff G/H$ is affine

H reductive $\iff U/H$ is affine

\uparrow always q -proj.

(Alper has proved the same result for lin. reductive fppf group schemes over arbitrary base)

Then choose $G \hookrightarrow GL_n$, then GL_n acts ^{transitively} on $X = GL_n/G = \text{Spec } A$
by above, have closed imm. $\text{Spec}(A) \hookrightarrow V$ for some GL_n -rep V

$\implies \forall$ reductive G , \exists a rep of GL_n and a $v \in V$ w/ closed orbit such that $G = \text{stab}(v)$

This statement (except for v having closed orbit) is true for any G .

Eg: an affine scheme w/ T action, where T is split torus, is equiv. to an M -graded algebra, where M is char lattice
 \hookrightarrow useful fact: every invariant ideal is gen. by homogeneous elements.

Next level of complexity: V linear rep of $G \rightsquigarrow$ action on $\mathbb{P}(V)$

Consider $X \hookrightarrow \mathbb{P}(V)$ which is equivariant $\left(G \times X \rightarrow \mathbb{P}(V) \text{ factors through } X, \text{ in which case it does so uniquely and induces an action on } X \right)$

(Rem: we will see later that this is true for any normal proj. variety)

Special results for T -varieties:

Thm (Sumihiro) Let $X \hookrightarrow \mathbb{P}(V)$ be a T -equivariant g -proj. subvariety, then for any point $x \in X$, \exists a T -equivariant affine open containing x .

Proof: Case 1: $X \hookrightarrow \mathbb{P}(V)$ closed: T splits, and $f \in V^*$ is eigenvect. use standard open $\mathbb{P}(V)_f \cap X$ where

Case 2: T is split, and $X \hookrightarrow \mathbb{P}(V)$ loc. closed: consider closure, then have T -inv. affine variety $Y \supset X \cap Y \ni x$

show that you can find an eigenvector $f \in \mathcal{O}(Y)$ which vanishes on $Y \setminus X \cap Y$ and does not vanish at x

General case: T splits after some finite Galois extension k'/k . Choose a $T_{k'}$ -equivariant affine open $U \subset X_{k'} \rightsquigarrow$ because X is separated,

$U = \bigcap_{\sigma \in \text{Gal}(k'/k)} \sigma(U)$ is a $T_{k'}$ -equiv. aff. open which loc. cont

$$U = \bigcup_{\sigma \in \text{Gal}(k'/k)} \sigma(U)$$

is k' -equiv. aff.
open, which descends
to a T -equiv. aff.
open of X



[INSERT DISCUSSION OF FIXED POINTS AND BIALYNICKI-BIRULA THEOREM]

[INSERT DISCUSSION OF QUOTIENTS]